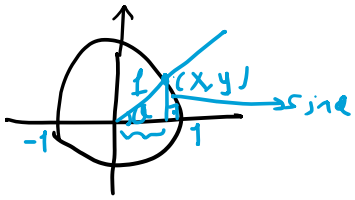
Trigonometric Identities

$x = \cos\theta$
$y = \sin\theta$



$$\cos^2\theta + \sin^2\theta = 1. \tag{3}$$

This equation, true for all values of θ, is the most frequently used identity in trigonometry. Dividing this identity in turn by $\cos^2\theta$ and $\sin^2\theta$ gives

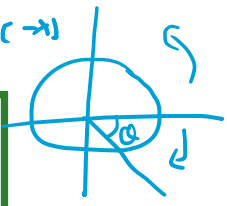
$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

$f(x) = f(-x) \rightarrow$ even function

$f(x) = -f(-x) \rightarrow$ odd function

$\cos(x) = \cos(-x)$

$\sin(x) = -\sin(-x)$



Addition Formulas

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$
$$\sin(A + B) = \sin A \cos B + \cos A \sin B \tag{4}$$

$\cos(A-B) = \cos(A+(-B)) = \cos A \cdot \cos(-B) - \sin A \cdot \sin(-B) = \cos A \cdot \cos B + \sin A \cdot \sin B$

Double-Angle Formulas

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\sin 2\theta = 2\sin\theta\cos\theta \tag{5}$$

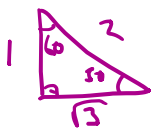
Half-Angle Formulas

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2} \tag{6}$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2} \tag{7}$$

$\cos 60 = \sin 30 = \frac{1}{2}$

$\sin 60 = \cos 30 = \frac{\sqrt{3}}{2}$



* Evaluate $\sin\frac{7\pi}{12}$ as $\left(\frac{\pi}{4} + \frac{\pi}{3}\right)$.

$$\sin\frac{7\pi}{12} = \sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right) = \sin\frac{\pi}{4}\cdot\cos\frac{\pi}{3} + \sin\frac{\pi}{3}\cdot\cos\frac{\pi}{4}$$
$$= \frac{\sqrt{2}}{2}\cdot\frac{1}{2} + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2} + \sqrt{6}}{4}$$

✳ Evaluate $\cos\frac{\pi}{12}$.

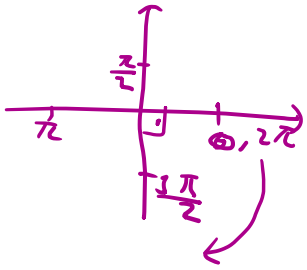
$\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$

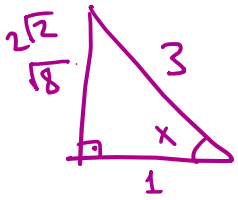
$\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{3} + \left(-\frac{\pi}{4}\right)\right)$

$= \cos\left(\frac{\pi}{3}\right)\cdot\cos\left(-\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{3}\right)\cdot\sin\left(-\frac{\pi}{4}\right)$

$= \frac{1}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\left(-\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}+\sqrt{6}}{4}$

✳ $\cos x = \frac{1}{3}$, $x \in [-\frac{\pi}{2}, 0]$ Find $\sin x$, $\tan x$.





$\sin x = -\frac{2\sqrt{2}}{3}$ $\quad \tan x = -\frac{2\sqrt{2}}{1}$

✳ Solve for the angle θ, where $0 \le \theta \le 2\pi$.

a) $\sin^2\theta = \frac{3}{4} \Rightarrow \sin\theta = \pm\sqrt{\frac{3}{4}}$

$\sin(\pi - \frac{\pi}{3}) = \sin(\frac{\pi}{3})$

$\sin\theta = +\frac{\sqrt{3}}{2} \rightarrow \theta = \frac{\pi}{3}, \pi - \frac{\pi}{3}$ ($\frac{2\pi}{3}$)

$\sin\theta = -\frac{\sqrt{3}}{2} \rightarrow \theta = \pi + \frac{\pi}{3}, 2\pi - \frac{\pi}{3}$ ($\frac{4\pi}{3}$, $\frac{5\pi}{3}$)

b) $\sin^2\theta = \cos^2\theta$

$\sqrt{\sin^2\theta} = \sqrt{\cos^2\theta} \Rightarrow \sin\theta = \cos\theta \Rightarrow \frac{\sin\theta}{\cos\theta} = 1 \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}, \pi + \frac{\pi}{4}$ ($\frac{5\pi}{4}$)

$\sin\theta = -\cos\theta \Rightarrow \frac{\sin\theta}{\cos\theta} = -1 \Rightarrow \tan\theta = -1 \Rightarrow \theta = \pi - \frac{\pi}{4}, 2\pi - \frac{\pi}{4}$ ($\frac{3\pi}{4}$, $\frac{7\pi}{4}$)

$\{\frac{\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}\}$

Two Special Inequalities

For any angle θ measured in radians,

$$-|\theta| \le \sin\theta \le |\theta| \qquad \text{and} \qquad -|\theta| \le 1 - \cos\theta \le |\theta|.$$

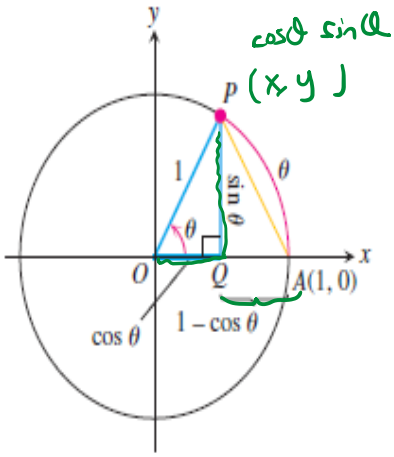


FIGURE 1.49 From the geometry of this figure, drawn for $\theta > 0$, we get the inequality $\sin^2\theta + (1 - \cos\theta)^2 \le \theta^2$.

To establish these inequalities, we picture θ as a nonzero angle in standard position (Figure 1.49). The circle in the figure is a unit circle, so $|\theta|$ equals the length of the circular arc AP. The length of line segment AP is therefore less than $|\theta|$.

Triangle APQ is a right triangle with sides of length

$$QP = |\sin\theta|, \qquad AQ = 1 - \cos\theta.$$

From the Pythagorean theorem and the fact that $AP < |\theta|$, we get

$$\sin^2\theta + (1 - \cos\theta)^2 = (AP)^2 \le \theta^2. \tag{9}$$

The terms on the left-hand side of Equation (9) are both positive, so each is smaller than their sum and hence is less than or equal to θ^2:

$$\sin^2\theta \le \theta^2 \qquad \text{and} \qquad (1 - \cos\theta)^2 \le \theta^2.$$

By taking square roots, this is equivalent to saying that

$$|\sin\theta| \le |\theta| \qquad \text{and} \qquad |1 - \cos\theta| \le |\theta|,$$

so

$$-|\theta| \le \sin\theta \le |\theta| \qquad \text{and} \qquad -|\theta| \le 1 - \cos\theta \le |\theta|.$$

These inequalities will be useful in the next chapter.

1.5 Exponential Functions

Exponential Behavior

When a positive quantity P doubles, it increases by a factor of 2 and the quantity becomes $2P$. If it doubles again, it becomes $2(2P) = 2^2P$, and a third doubling gives $2(2^2P) = 2^3P$. Continuing to double in this fashion leads us to the consideration of the function $f(x) = 2^x$. We call this an *exponential* function because the variable x appears in the exponent of 2^x. Functions such as $g(x) = 10^x$ and $h(x) = (1/2)^x$ are other examples of exponential functions. In general, if $a \neq 1$ is a positive constant, the function

$$f(x) = a^x$$

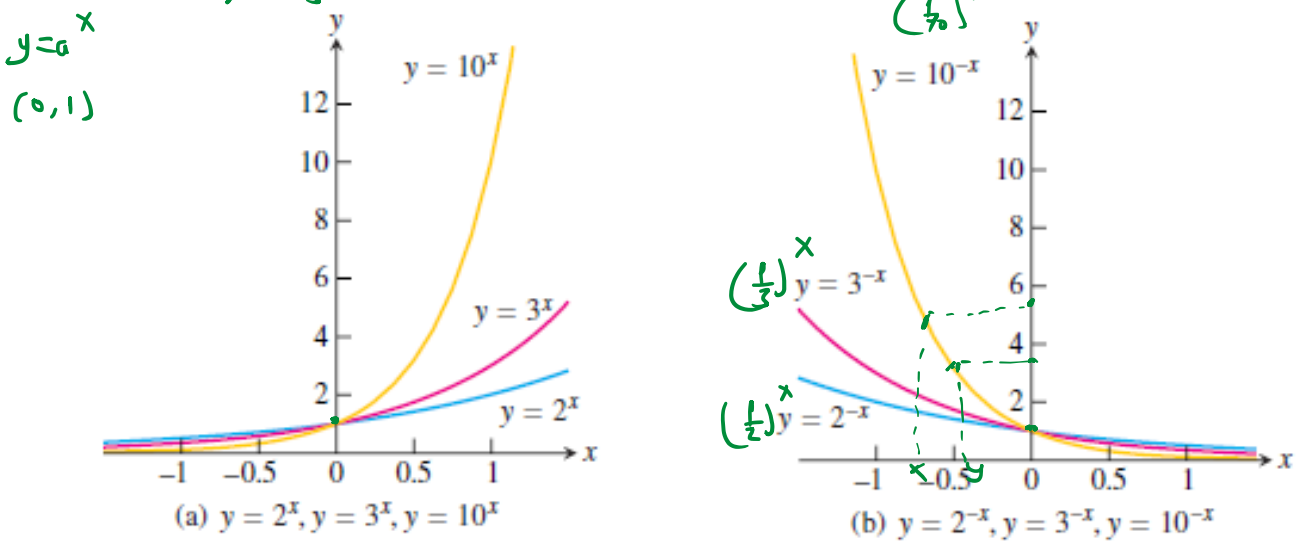
is the **exponential function with base a.**

For integer and rational exponents, the value of an exponential function $f(x) = a^x$ is obtained arithmetically as follows. If $x = n$ is a positive integer, the number a^n is given by multiplying a by itself n times:

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}.$$

If $x = 0$, then $a^0 = 1$, and if $x = -n$ for some positive integer n, then

$$a^{-n} = \frac{1}{a^n} = \left(\frac{1}{a}\right)^n$$



(a) $y = 2^x, y = 3^x, y = 10^x$ (b) $y = 2^{-x}, y = 3^{-x}, y = 10^{-x}$

If $x = 1/n$ for some positive integer n, then

$$a^{1/n} = \sqrt[n]{a},$$

which is the positive number that when multiplied by itself n times gives a. If $x = p/q$ is any rational number, then

$$a^{p/q} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p.$$

Rules for Exponents

If $a > 0$ and $b > 0$, the following rules hold true for all real numbers x and y.

1. $a^x \cdot a^y = a^{x+y}$
2. $\dfrac{a^x}{a^y} = a^{x-y}$
3. $(a^x)^y = (a^y)^x = a^{xy}$
4. $a^x \cdot b^x = (ab)^x$
5. $\dfrac{a^x}{b^x} = \left(\dfrac{a}{b}\right)^x$

The Natural Exponential Function e^x

The most important exponential function used for modeling natural, physical, and economic phenomena is the **natural exponential function**, whose base is the special number e. The number e is irrational, and its value is 2.718281828 to nine decimal places. It might seem strange that we would use this number for a base rather than a simple number like 2 or 10. The advantage in using e as a base is that it simplifies many of the calculations in calculus.

If you look at Figure 1.56a you can see that the graphs of the exponential functions $y = a^x$ get steeper as the base a gets larger. This idea of steepness is conveyed by the slope of the tangent line to the graph at a point. Tangent lines to graphs of functions are defined precisely in the next chapter, but intuitively the tangent line to the graph at a point is a line that just touches the graph at the point, like a tangent to a circle. Figure 1.57 shows the slope of the graph of $y = a^x$ as it crosses the y-axis for several values of a. Notice that the slope is exactly equal to 1 when a equals the number e. The slope is smaller than 1 if $a < e$, and larger than 1 if $a > e$. This is the property that makes the number e so useful in calculus: **The graph of $y = e^x$ has slope 1 when it crosses the y-axis.**

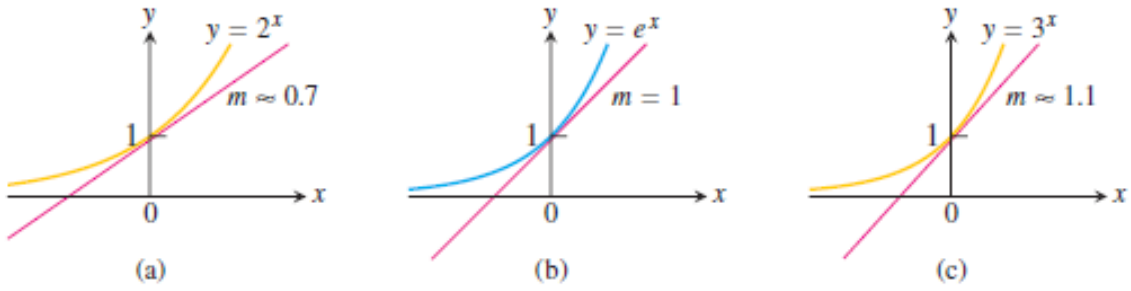


FIGURE 1.57 Among the exponential functions, the graph of $y = e^x$ has the property that the slope m of the tangent line to the graph is exactly 1 when it crosses the y-axis. The slope is smaller for a base less than e, such as 2^x, and larger for a base greater than e, such as 3^x.

In Chapter 3 we use that slope property to prove e is the number the quantity $(1 + 1/x)^x$ approaches as x becomes large without bound. That result provides one way to compute the value of e, at least approximately. The graph and table in Figure 1.58 show the behavior of this expression and how it gets closer and closer to the line $y = e \approx 2.718281828$ as x gets larger and larger. (This *limit* idea is made precise in the next chapter.) A more complete discussion of e is given in Chapter 7.

x	$(1 + 1/x)^x$
1000	2.7169
2000	2.7176
3000	2.7178
4000	2.7179
5000	2.7180
6000	2.7181
7000	2.7181

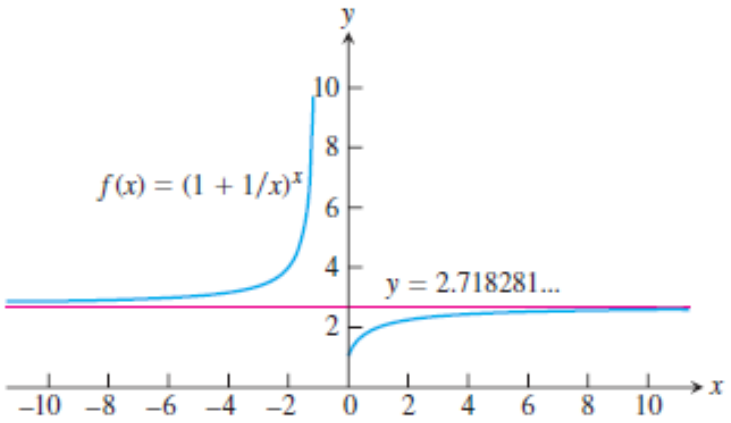


FIGURE 1.58 A graph and table of values for $f(x) = (1 + 1/x)^x$ both suggest that as x gets larger and larger, $f(x)$ gets closer and closer to $e \approx 2.7182818\ldots$.

1.6 Inverse Functions and Logarithms

A function that undoes, or inverts, the effect of a function f is called the *inverse* of f. Many common functions, though not all, are paired with an inverse. In this section we present the natural logarithmic function $y = \ln x$ as the inverse of the exponential function $y = e^x$, and we also give examples of several inverse trigonometric functions.

One-to-One Functions

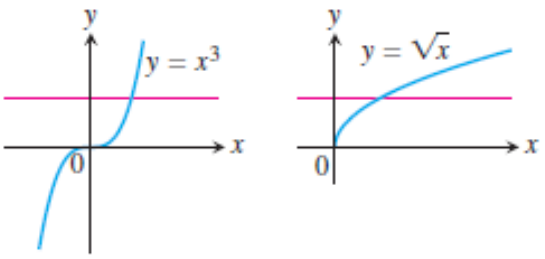
A function is a rule that assigns a value from its range to each element in its domain. Some functions assign the same range value to more than one element in the domain. The function $f(x) = x^2$ assigns the same value, 1, to both of the numbers -1 and $+1$; the sines of $\pi/3$ and $2\pi/3$ are both $\sqrt{3}/2$. Other functions assume each value in their range no more than once. The square roots and cubes of different numbers are always different. A function that has distinct values at distinct elements in its domain is called one-to-one. These functions take on any one value in their range exactly once.

DEFINITION A function $f(x)$ is **one-to-one** on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

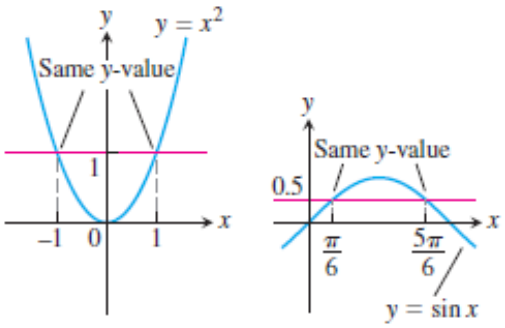
The graph of a one-to-one function $y = f(x)$ can intersect a given horizontal line at most once. If the function intersects the line more than once, it assumes the same y-value for at least two different x-values and is therefore not one-to-one (Figure 1.60).

The Horizontal Line Test for One-to-One Functions
A function $y = f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.

$1 \neq -1 \quad f(1) = f(-1)$



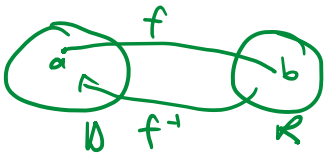
(a) One-to-one: Graph meets each horizontal line at most once.



(b) Not one-to-one: Graph meets one or more horizontal lines more than once.

$y = x^2$ is a one to one function for $x \geq 0$ $[0, \infty)$

Inverse Functions



Since each output of a one-to-one function comes from just one input, the effect of the function can be inverted to send an output back to the input from which it came.

DEFINITION Suppose that f is a one-to-one function on a domain D with range R. The **inverse function** f^{-1} is defined by

$$f^{-1}(b) = a \text{ if } f(a) = b.$$

The domain of f^{-1} is R and the range of f^{-1} is D.

The symbol f^{-1} for the inverse of f is read "f inverse." The "-1" in f^{-1} is *not* an exponent; $f^{-1}(x)$ does not mean $1/f(x)$. Notice that the domains and ranges of f and f^{-1} are interchanged.

If we apply f to send an input x to the output $f(x)$ and follow by applying f^{-1} to $f(x)$ we get right back to x, just where we started. Similarly, if we take some number y in the range of f, apply f^{-1} to it, and then apply f to the resulting value $f^{-1}(y)$, we get back the value y with which we began. Composing a function and its inverse has the same effect as doing nothing.

$$(f^{-1} \circ f)(x) = x, \quad \text{for all } x \text{ in the domain of } f$$

$$(f \circ f^{-1})(y) = y, \quad \text{for all } y \text{ in the domain of } f^{-1} \text{ (or range of } f)$$

Finding Inverses

The process of passing from f to f^{-1} can be summarized as a two-step procedure.

1. Solve the equation $y = f(x)$ for x. This gives a formula $x = f^{-1}(y)$ where x is expressed as a function of y.
2. Interchange x and y, obtaining a formula $y = f^{-1}(x)$ where f^{-1} is expressed in the conventional format with x as the independent variable and y as the dependent variable.

EXAMPLE 3 Find the inverse of $y = \frac{1}{2}x + 1$, expressed as a function of x.

Solution

1. *Solve for x in terms of y:* $y = \frac{1}{2}x + 1$

$$2y = x + 2$$

$$x = 2y - 2.$$

2. *Interchange x and y:* $y = 2x - 2.$

The inverse of the function $f(x) = (1/2)x + 1$ is the function $f^{-1}(x) = 2x - 2$. (See Figure 1.62.) To check, we verify that both composites give the identity function:

$$f^{-1}(f(x)) = 2\left(\frac{1}{2}x + 1\right) - 2 = x + 2 - 2 = x$$

$$f(f^{-1}(x)) = \frac{1}{2}(2x - 2) + 1 = x - 1 + 1 = x.$$ ■

EXAMPLE 4 Find the inverse of the function $y = x^2, x \geq 0$, expressed as a function of x.

Solution We first solve for x in terms of y:

$$y = x^2$$

$$\sqrt{y} = \sqrt{x^2} = |x| = x \qquad |x| = x \text{ because } x \geq 0$$

2) We then interchange x and y, obtaining

$$y = \sqrt{x}.$$

The inverse of the function $y = x^2, x \geq 0$, is the function $y = \sqrt{x}$ (Figure 1.63).

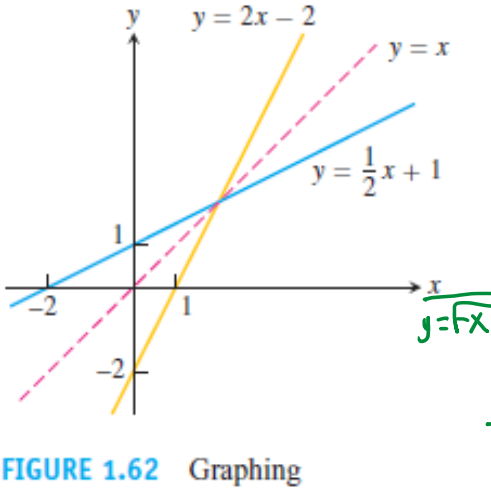


FIGURE 1.62 Graphing $f(x) = (1/2)x + 1$ and $f^{-1}(x) = 2x - 2$ together shows the graphs' symmetry with respect to the line $y = x$ (Example 3).

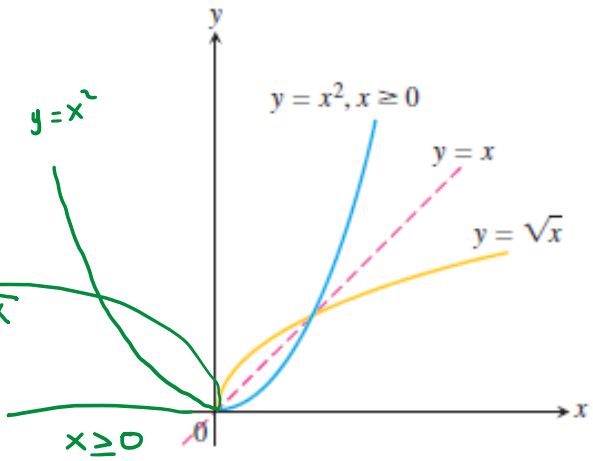


FIGURE 1.63 The functions $y = \sqrt{x}$ and $y = x^2, x \geq 0$, are inverses of one another (Example 4).

D(f) = (−∞,∞)
R(f) = (0,∞)
$y = a^x$
1) $x = \log_a y$
2) $y = \log_a x$
D(f) = (0,∞)
R(f) = (−∞,∞)

Logarithmic Functions

If a is any positive real number other than 1, the base a exponential function $f(x) = a^x$ is one-to-one. It therefore has an inverse. Its inverse is called the *logarithm function with base a*.

DEFINITION The **logarithm function with base** a, $y = \log_a x$, is the inverse of the base a exponential function $y = a^x$ $(a > 0, a \neq 1)$.

$\log_e x$ is written as $\ln x$.
$\log_{10} x$ is written as $\log x$.

The function $y = \ln x$ is called the **natural logarithm function**, and $y = \log x$ is often called the **common logarithm function**. For the natural logarithm,

$$\ln x = y \iff e^y = x.$$

In particular, if we set $x = e$, we obtain

$$\ln e = 1$$

because $e^1 = e$.

THEOREM 1—Algebraic Properties of the Natural Logarithm For any numbers $b > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

1. *Product Rule*: $\ln bx = \ln b + \ln x$
2. *Quotient Rule*: $\ln \frac{b}{x} = \ln b - \ln x$
3. *Reciprocal Rule*: $\ln \frac{1}{x} = -\ln x$ Rule 2 with $b = 1$
4. *Power Rule*: $\ln x^r = r \ln x$

Inverse Properties for a^x and $\log_a x$

1. Base a: $a^{\log_a x} = x$, $\log_a a^x = x$, $a > 0, a \neq 1, x > 0$
2. Base e: $e^{\ln x} = x$, $\ln e^x = x$, $x > 0$

Substituting a^x for x in the equation $x = e^{\ln x}$ enables us to rewrite a^x as a power of e:

$$\begin{aligned} a^x &= e^{\ln(a^x)} && \text{Substitute } a^x \text{ for } x \text{ in } x = e^{\ln x}. \\ &= e^{x \ln a} && \text{Power Rule for logs} \\ &= e^{(\ln a)x}. && \text{Exponent rearranged} \end{aligned}$$

Thus, the exponential function a^x is the same as e^{kx} for $k = \ln a$.

> Every exponential function is a power of the natural exponential function.
>
> $$a^x = e^{x \ln a}$$
>
> That is, a^x is the same as e^x raised to the power $\ln a$: $a^x = e^{kx}$ for $k = \ln a$.

$x = \ln a^x$

For example,

$$2^x = e^{(\ln 2)x} = e^{x \ln 2}, \quad \text{and} \quad 5^{-3x} = e^{(\ln 5)(-3x)} = e^{-3x \ln 5}.$$

Returning once more to the properties of a^x and $\log_a x$, we have

$$\begin{aligned} \ln x &= \ln(a^{\log_a x}) && \text{Inverse Property for } a^x \text{ and } \log_a x \\ &= (\log_a x)(\ln a). && \text{Power Rule for logarithms, with } r = \log_a x \end{aligned}$$

Rewriting this equation as $\log_a x = (\ln x)/(\ln a)$ shows that every logarithmic function is a constant multiple of the natural logarithm $\ln x$. This allows us to extend the algebraic properties for $\ln x$ to $\log_a x$. For instance, $\log_a bx = \log_a b + \log_a x$.

> **Change of Base Formula**
>
> Every logarithmic function is a constant multiple of the natural logarithm.
>
> $$\log_a x = \frac{\ln x}{\ln a} \qquad (a > 0, a \neq 1)$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

* Simplify the given expressions.

a) $\log_2\left(e^{(\ln 2)(\sin x)}\right) = \log_2\left(e^{\ln 2}\right)^{\sin x} = \log_2(2)^{\sin x} = \sin x \cdot \underbrace{\log_2 2}_{=1} = \sin x$

b) $\ln\left(e^{(e^x)}\right) = \ln e^{e^x} = e^x \underbrace{\ln e}_{=1} = e^x$

c) $\dfrac{(a^5)^2 a^{-6}}{\sqrt[3]{a^2}}$

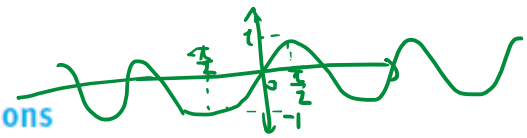
✳ Solve for y

a) $\ln(y^2-1) - \ln(y+1) = \ln(\sin x)$

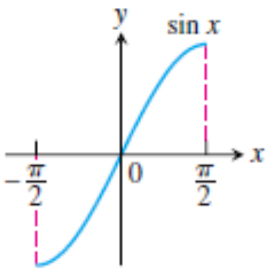
$\ln\left(\frac{y^2-1}{y+1}\right) = \ln(\sin x) \Rightarrow \ln\left(\frac{(y-1)(y+1)}{(y+1)}\right) = \ln(\sin x)$

$\ln(y-1) = \ln(\sin x) \Rightarrow y-1 = \sin x \Rightarrow y = \sin x + 1$

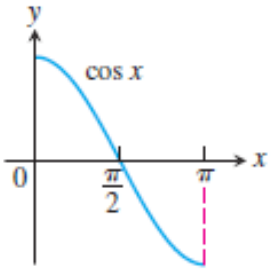
b) $\ln(y-1) - \ln 2 = x + \ln x$



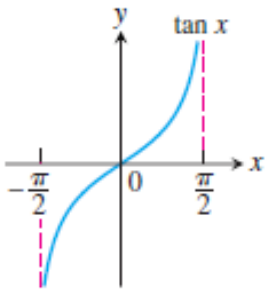
Domain restrictions that make the trigonometric functions one-to-one



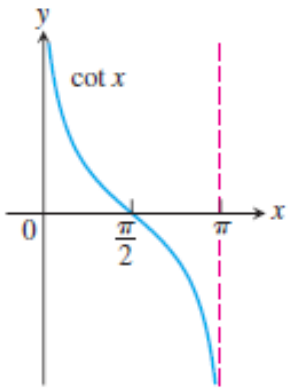
$y = \sin x$
Domain: $[-\pi/2, \pi/2]$
Range: $[-1, 1]$



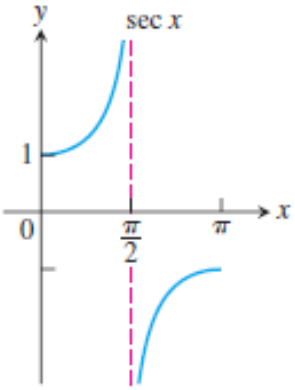
$y = \cos x$
Domain: $[0, \pi]$
Range: $[-1, 1]$



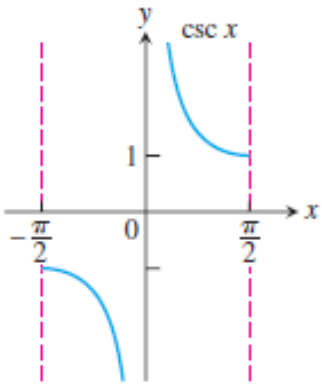
$y = \tan x$
Domain: $(-\pi/2, \pi/2)$
Range: $(-\infty, \infty)$



$y = \cot x$
Domain: $(0, \pi)$
Range: $(-\infty, \infty)$



$y = \sec x$
Domain: $[0, \pi/2) \cup (\pi/2, \pi]$
Range: $(-\infty, -1] \cup [1, \infty)$



$y = \csc x$
Domain: $[-\pi/2, 0) \cup (0, \pi/2]$
Range: $(-\infty, -1] \cup [1, \infty)$

Since these restricted functions are now one-to-one, they have inverses, which we denote by

$$\begin{aligned}
y &= \sin^{-1} x \quad \text{or} \quad y = \arcsin x \\
y &= \cos^{-1} x \quad \text{or} \quad y = \arccos x \\
y &= \tan^{-1} x \quad \text{or} \quad y = \arctan x \\
y &= \cot^{-1} x \quad \text{or} \quad y = \operatorname{arccot} x \\
y &= \sec^{-1} x \quad \text{or} \quad y = \operatorname{arcsec} x \\
y &= \csc^{-1} x \quad \text{or} \quad y = \operatorname{arccsc} x
\end{aligned}$$

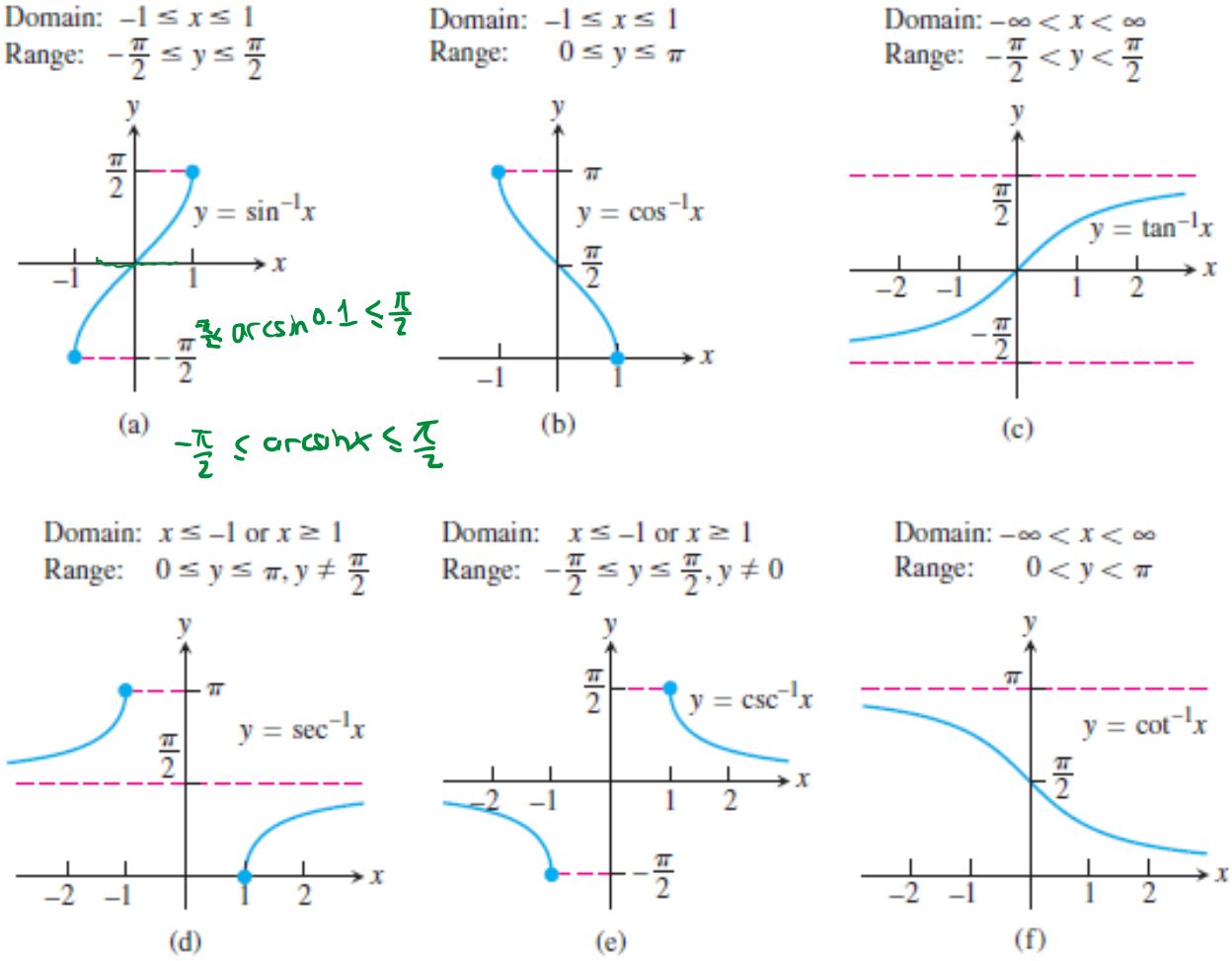

Domain: $-1 \le x \le 1$
Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \sin^{-1}x$
(a)
Domain: $-1 \le x \le 1$
Range: $0 \le y \le \pi$
$y = \cos^{-1}x$
(b)
Domain: $-\infty < x < \infty$
Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = \tan^{-1}x$
(c)
Domain: $x \le -1$ or $x \ge 1$
Range: $0 \le y \le \pi, y \ne \frac{\pi}{2}$
$y = \sec^{-1}x$
(d)
Domain: $x \le -1$ or $x \ge 1$
Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}, y \ne 0$
$y = \csc^{-1}x$
(e)
Domain: $-\infty < x < \infty$
Range: $0 < y < \pi$
$y = \cot^{-1}x$
(f)

FIGURE 1.66 Graphs of the six basic inverse trigonometric functions.

DEFINITION

$y = \sin^{-1}x$ is the number in $[-\pi/2, \pi/2]$ for which $\sin y = x$.

$y = \cos^{-1}x$ is the number in $[0, \pi]$ for which $\cos y = x$.

The graph of $y = \sin^{-1}x$ (Figure 1.67b) is symmetric about the origin (it lies along the graph of $x = \sin y$). The arcsine is therefore an odd function:

$$\sin^{-1}(-x) = -\sin^{-1}x. \tag{2}$$

The graph of $y = \cos^{-1}x$ (Figure 1.68b) has no such symmetry.

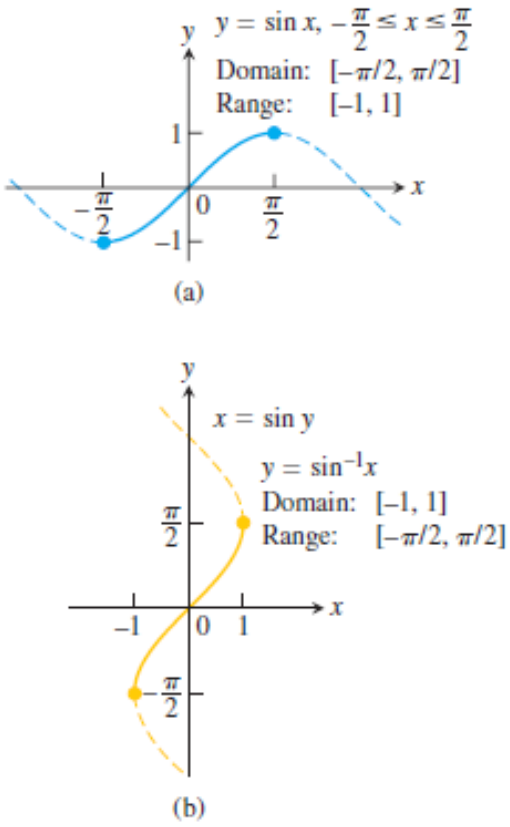


FIGURE 1.67 The graphs of (a) $y = \sin x, -\pi/2 \le x \le \pi/2$, and (b) its inverse, $y = \sin^{-1} x$. The graph of $\sin^{-1} x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \sin y$.

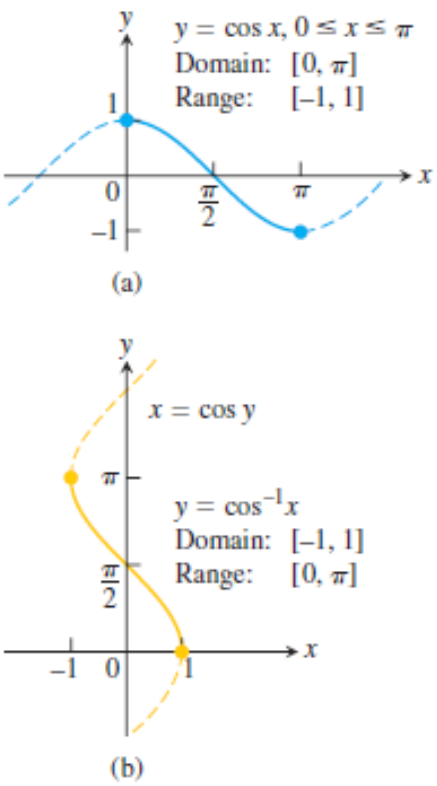


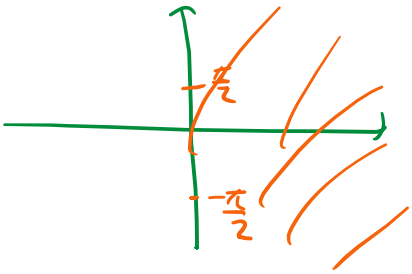
FIGURE 1.68 The graphs of (a) $y = \cos x, 0 \le x \le \pi$, and (b) its inverse, $y = \cos^{-1} x$. The graph of $\cos^{-1} x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \cos y$.

EXAMPLE 8 Evaluate **(a)** $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ and **(b)** $\cos^{-1}\left(-\frac{1}{2}\right)$.

(a) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = x \quad , x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\sin x = \frac{\sqrt{3}}{2}$

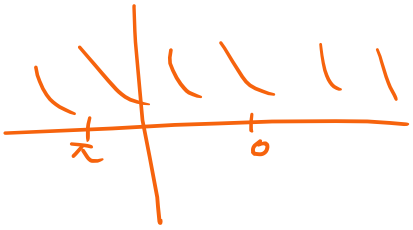
$x = \frac{\pi}{3}$



(b) $\cos^{-1}\left(-\frac{1}{2}\right) = x \quad , x \in [0, \pi]$

$\cos x = -\frac{1}{2}$

$x = \pi - \frac{\pi}{3} \Rightarrow x = \frac{2\pi}{3}$



Using the same procedure illustrated in Example 8, we can create the following table of common values for the arcsine and arccosine functions.

x	$\sin^{-1} x$	$\cos^{-1} x$
$\sqrt{3}/2$	$\pi/3$	$\pi/6$
$\sqrt{2}/2$	$\pi/4$	$\pi/4$
$1/2$	$\pi/6$	$\pi/3$
$-1/2$	$-\pi/6$	$2\pi/3$
$-\sqrt{2}/2$	$-\pi/4$	$3\pi/4$
$-\sqrt{3}/2$	$-\pi/3$	$5\pi/6$

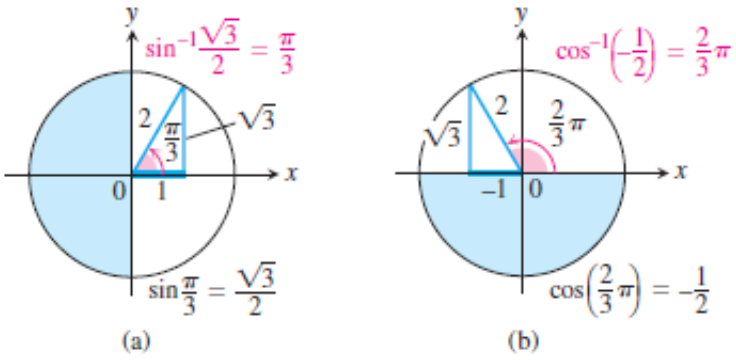


FIGURE 1.69 Values of the arcsine and arccosine functions (Example 8).

2.1 Rates of Change and Tangents to Curves

Average and Instantaneous Speed

In the late sixteenth century, Galileo discovered that a solid object dropped from rest (not moving) near the surface of the earth and allowed to fall freely will fall a distance proportional to the square of the time it has been falling. This type of motion is called **free fall**. It assumes negligible air resistance to slow the object down, and that gravity is the only force acting on the falling body. If y denotes the distance fallen in feet after t seconds, then Galileo's law is

$$y = 16t^2,$$

where 16 is the (approximate) constant of proportionality. (If y is measured in meters, the constant is 4.9.)

A moving body's **average speed** during an interval of time is found by dividing the distance covered by the time elapsed. The unit of measure is length per unit time:
kilometers per hour, feet (or meters) per second, or whatever is appropriate to the problem at hand.

EXAMPLE 1 A rock breaks loose from the top of a tall cliff. What is its average speed

(a) during the first 2 sec of fall?

(b) during the 1-sec interval between second 1 and second 2?

Solution The average speed of the rock during a given time interval is the change in distance, Δy, divided by the length of the time interval, Δt. (Increments like Δy and Δt are reviewed in Appendix 3.) Measuring distance in feet and time in seconds, we have the following calculations:

(a) For the first 2 sec:
$$\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(0)^2}{2 - 0} = 32\,\frac{\text{ft}}{\text{sec}}$$

(b) From sec 1 to sec 2:
$$\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(1)^2}{2 - 1} = 48\,\frac{\text{ft}}{\text{sec}}$$

■

EXAMPLE 2 Find the speed of the falling rock in Example 1 at $t = 1$ and $t = 2$ sec.

Solution We can calculate the average speed of the rock over a time interval $[t_0, t_0 + h]$, having length $\Delta t = h$, as

$$\frac{\Delta y}{\Delta t} = \frac{16(t_0 + h)^2 - 16{t_0}^2}{h}. \qquad (1)$$

We cannot use this formula to calculate the "instantaneous" speed at the exact moment t_0 by simply substituting $h = 0$, because we cannot divide by zero. But we *can* use it to calculate average speeds over increasingly short time intervals starting at $t_0 = 1$ and $t_0 = 2$. When we do so, we see a pattern (Table 2.1).

TABLE 2.1 Average speeds over short time intervals $[t_0, t_0 + h]$

Average speed: $\dfrac{\Delta y}{\Delta t} = \dfrac{16(t_0 + h)^2 - 16t_0^2}{h}$

Length of time interval h	Average speed over interval of length h starting at $t_0 = 1$	Average speed over interval of length h starting at $t_0 = 2$
1	48	80
0.1	33.6	65.6
0.01	32.16	64.16
0.001	32.016	64.016
0.0001	32.0016	64.0016

The average speed on intervals starting at $t_0 = 1$ seems to approach a limiting value of 32 as the length of the interval decreases. This suggests that the rock is falling at a speed of 32 ft/sec at $t_0 = 1$ sec. Let's confirm this algebraically.

If we set $t_0 = 1$ and then expand the numerator in Equation (1) and simplify, we find that

$$\frac{\Delta y}{\Delta t} = \frac{16(1 + h)^2 - 16(1)^2}{h} = \frac{16(1 + 2h + h^2) - 16}{h}$$

$$= \frac{32h + 16h^2}{h} = 32 + 16h.$$

For values of h different from 0, the expressions on the right and left are equivalent and the average speed is $32 + 16h$ ft/sec. We can now see why the average speed has the limiting value $32 + 16(0) = 32$ ft/sec as h approaches 0.

Similarly, setting $t_0 = 2$ in Equation (1), the procedure yields

$$\frac{\Delta y}{\Delta t} = 64 + 16h$$

for values of h different from 0. As h gets closer and closer to 0, the average speed has the limiting value 64 ft/sec when $t_0 = 2$ sec, as suggested by Table 2.1. ■

Average Rates of Change and Secant Lines

Given an arbitrary function $y = f(x)$, we calculate the average rate of change of y with respect to x over the interval $[x_1, x_2]$ by dividing the change in the value of y, $\Delta y = f(x_2) - f(x_1)$, by the length $\Delta x = x_2 - x_1 = h$ of the interval over which the change occurs. (We use the symbol h for Δx to simplify the notation here and later on.)

DEFINITION The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \qquad h \neq 0.$$

Geometrically, the rate of change of f over $[x_1, x_2]$ is the slope of the line through the points $P(x_1, f(x_1))$ and $Q(x_2, f(x_2))$ (Figure 2.1).
In geometry, a line joining two points of a curve is a **secant** to the curve. Thus, the average rate of change of f from x_1 to x_2 is identical with the slope of secant PQ. Let's consider what happens as the point Q approaches the point P along the curve, so the length h of the interval over which the change occurs approaches zero.

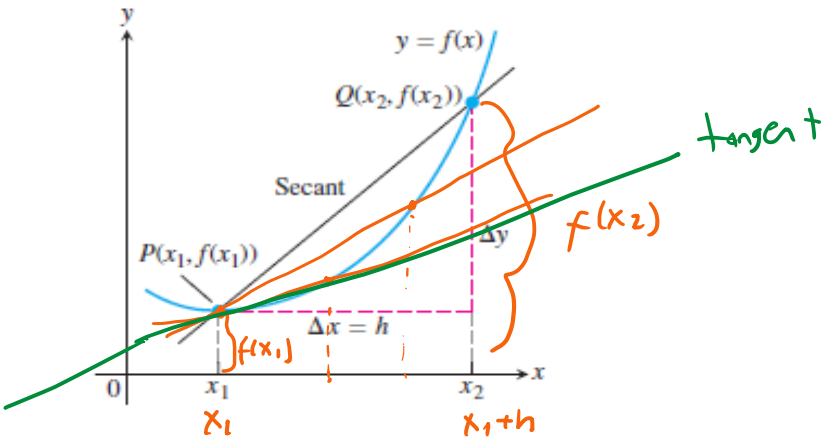


FIGURE 2.1 A secant to the graph $y = f(x)$. Its slope is $\Delta y/\Delta x$, the average rate of change of f over the interval $[x_1, x_2]$.

Defining the Slope of a Curve

We know what is meant by the slope of a straight line, which tells us the rate at which it rises or falls—its rate of change as the graph of a linear function. But what is meant by the *slope of a curve* at a point P on the curve? If there is a *tangent* line to the curve at P—a line that just touches the curve like the tangent to a circle—it would be reasonable to identify *the slope of the tangent* as the slope of the curve at P. So we need a precise meaning for the tangent at a point on a curve.

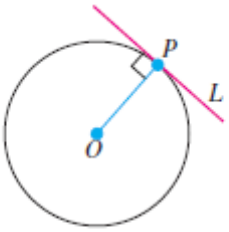


FIGURE 2.2 L is tangent to the circle at P if it passes through P perpendicular to radius OP.

To define tangency for general curves, we need an approach that takes into account the behavior of the secants through P and nearby points Q as Q moves toward P along the curve (Figure 2.3). Here is the idea:

1. Start with what we *can* calculate, namely the slope of the secant PQ.
2. Investigate the limiting value of the secant slope as Q approaches P along the curve. (We clarify the *limit* idea in the next section.)
3. If the *limit* exists, take it to be the slope of the curve at P and *define* the tangent to the curve at P to be the line through P with this slope.

This procedure is what we were doing in the falling-rock problem discussed in Example 2. The next example illustrates the geometric idea for the tangent to a curve.

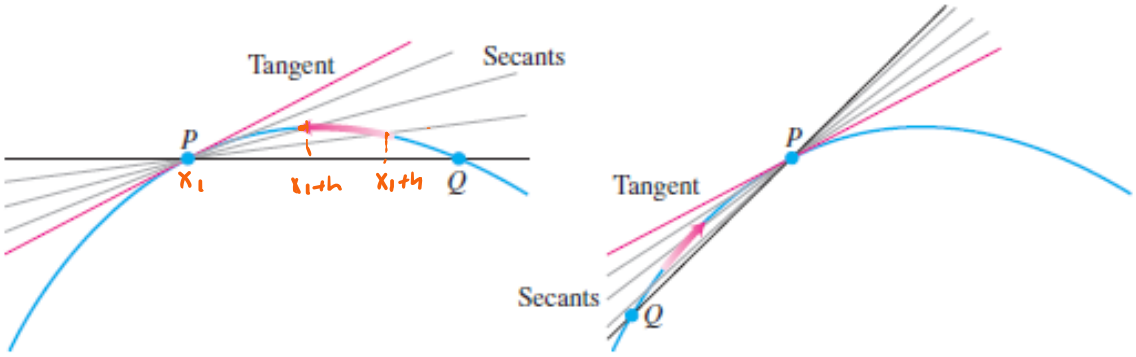


FIGURE 2.3 The tangent to the curve at P is the line through P whose slope is the limit of the secant slopes as $Q \to P$ from either side.

$P(x_0, y_0)$ m = slope $\quad y - y_0 = m(x - x_0)$ ↘ line equation

EXAMPLE 3 Find the slope of the parabola $y = x^2$ at the point $P(2, 4)$. Write an equation for the tangent to the parabola at this point. $f(x) = x^2$

Solution We begin with a secant line through $P(2, 4)$ and $Q(2 + h, (2 + h)^2)$ nearby. We then write an expression for the slope of the secant PQ and investigate what happens to the slope as Q approaches P along the curve:

$\frac{f(2+h) - f(2)}{h}$

$$\text{Secant slope} = \frac{\Delta y}{\Delta x} = \frac{(2 + h)^2 - 2^2}{h} = \frac{h^2 + 4h + 4 - 4}{h}$$

$h(h+4)$

$$= \frac{h^2 + 4h}{h} = h + 4.$$

If $h > 0$, then Q lies above and to the right of P, as in Figure 2.4. If $h < 0$, then Q lies to the left of P (not shown). In either case, as Q approaches P along the curve, h approaches zero and the secant slope $h + 4$ approaches 4. We take 4 to be the parabola's slope at P.

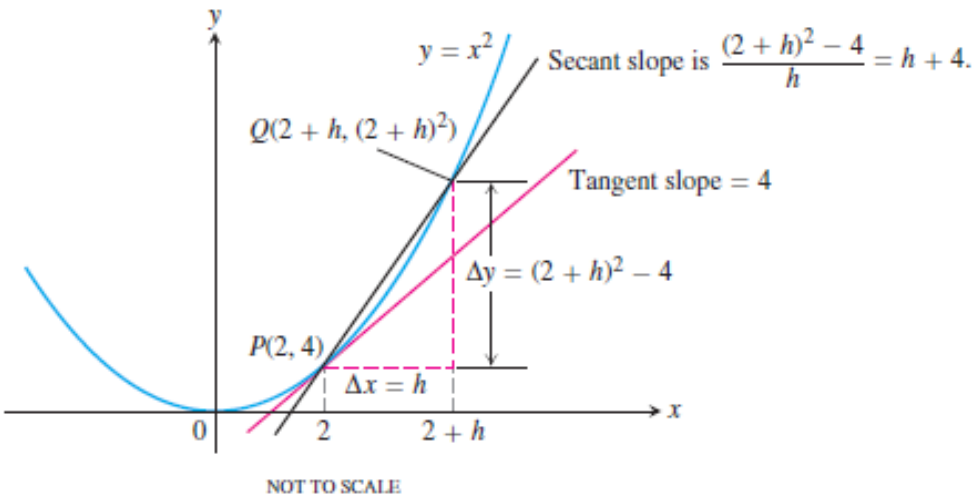


FIGURE 2.4 Finding the slope of the parabola $y = x^2$ at the point $P(2, 4)$ as the limit of secant slopes (Example 3).

The tangent to the parabola at P is the line through P with slope 4:

(2,4)
m = 4

$$y = 4 + 4(x - 2) \qquad \text{Point-slope equation}$$

$$y = 4x - 4.$$

$\lim_{h \to 0} \frac{f(x_1+h) - f(x_1)}{h}$ = slope of the tangent to $f(x)$ at the point x_1 $= f'(x_1)$

Limits of Function Values

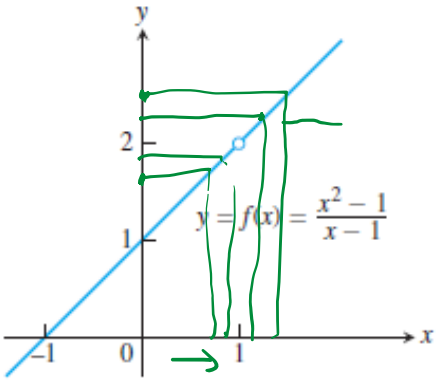
Frequently when studying a function $y = f(x)$, we find ourselves interested in the function's behavior *near* a particular point x_0, but not *at* x_0. This might be the case, for instance, if x_0 is an irrational number, like π or $\sqrt{2}$, whose values can only be approximated by "close" rational numbers at which we actually evaluate the function instead. Another situation occurs when trying to evaluate a function at x_0 leads to division by zero, which is undefined. We encountered this last circumstance when seeking the instantaneous rate of change in y by considering the quotient function $\Delta y/h$ for h closer and closer to zero. Here's a specific example where we explore numerically how a function behaves near a particular point at which we cannot directly evaluate the function.

EXAMPLE 1 How does the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

behave near $x = 1$?

$\lim_{x \to 1} f(x) = \lim_{\substack{x \to 1 \\ x \neq 1}} \frac{(x-1)(x+1)}{x-1} = \lim_{x \to 1} (x+1) = 2$



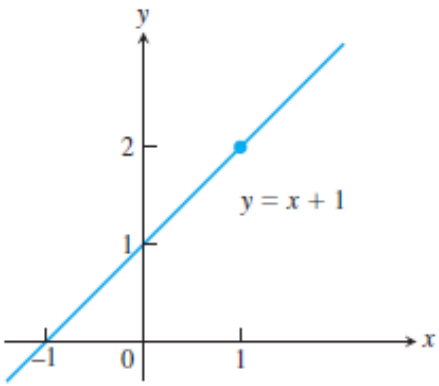


TABLE 2.2 The closer x gets to 1, the closer $f(x) = (x^2 - 1)/(x - 1)$ seems to get to 2

Values of x below and above 1	$f(x) = \dfrac{x^2 - 1}{x - 1} = x + 1, \quad x \neq 1$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001
0.999999	1.999999
1.000001	2.000001

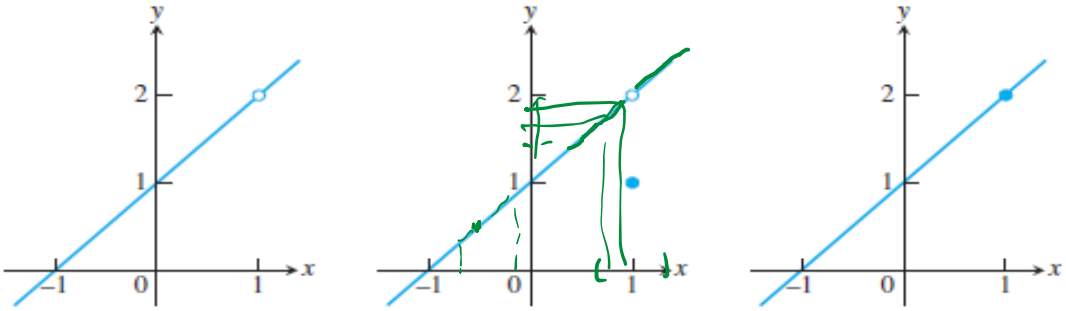
Suppose $f(x)$ is defined on an open interval about x_0, *except possibly at* x_0 *itself*. If $f(x)$ is arbitrarily close to L (as close to L as we like) for all x sufficiently close to x_0, we say that f approaches the **limit** L as x approaches x_0, and write

$$\lim_{x \to x_0} f(x) = L,$$

which is read "the limit of $f(x)$ as x approaches x_0 is L." For instance, in Example 1 we would say that $f(x)$ approaches the *limit* 2 as x approaches 1, and write

$$\lim_{x \to 1} f(x) = 2, \qquad \text{or} \qquad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2.$$

EXAMPLE 2 This example illustrates that the limit value of a function does not depend on how the function is defined at the point being approached. Consider the three functions in Figure 2.8. The function f has limit 2 as $x \to 1$ even though f is not defined at $x = 1$.



(a) $f(x) = \dfrac{x^2 - 1}{x - 1}$ (b) $g(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$ (c) $h(x) = x + 1$

$\lim_{x \to 1} g(x) = 2$

FIGURE 2.8 The limits of $f(x)$, $g(x)$, and $h(x)$ all equal 2 as x approaches 1. However, only $h(x)$ has the same function value as its limit at $x = 1$ (Example 2).

EXAMPLE 3

(a) If f is the **identity function** $f(x) = x$, then for any value of x_0 (Figure 2.9a),

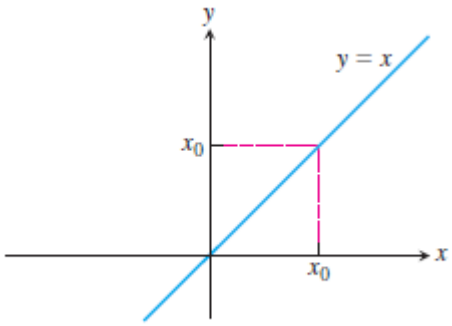
$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} x = x_0.$$

(b) If f is the **constant function** $f(x) = k$ (function with the constant value k), then for any value of x_0 (Figure 2.9b),

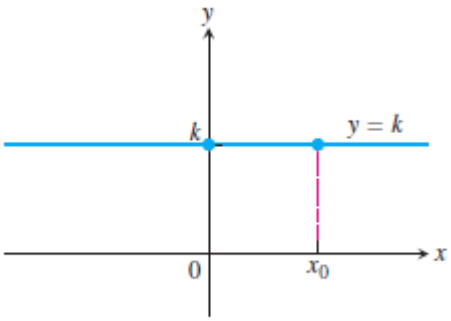
$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} k = k.$$

For instances of each of these rules we have

$$\lim_{x \to 3} x = 3 \qquad \text{and} \qquad \lim_{x \to -7}(4) = \lim_{x \to 2}(4) = 4.$$



(a) Identity function



(b) Constant function

EXAMPLE 4

Discuss the behavior of the following functions as $x \to 0$.

(a) $U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$

(b) $g(x) = \begin{cases} \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(c) $f(x) = \begin{cases} 0, & x \leq 0 \\ \sin \dfrac{1}{x}, & x > 0 \end{cases}$

$\lim_{x \to -3} \frac{x+3}{x^2+4x+3} = \frac{x+3}{(x+3)(x+1)} = \frac{1}{x+1}$ Birbirini götürür.

$\lim_{x \to 0} \frac{\frac{1}{x-1} + \frac{1}{x+1}}{x} = \frac{x+1+x-1}{(x-1)\cdot(x+1)} = \frac{2x}{(x-1)(x+1)} \cdot \frac{1}{x} = \frac{2}{x^2-1} = \frac{2}{-1} = -2$

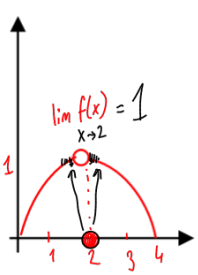
$\lim_{x \to 0} \frac{\sqrt{x^2+10^2}-10}{x^2} = \frac{\sqrt{x^2+100}+10}{\sqrt{x^2+100}+10}$

$x^2 - y^2 = (x-y)(x+y)$

$\lim_{x \to 0} 1 - \frac{x^2}{4} = 1$ (4)

$\frac{4}{4} - \frac{x^2}{4} = 1$

$\frac{4-x^2}{4} = 1$



$x \to 2$'nin limiti 1 dir.

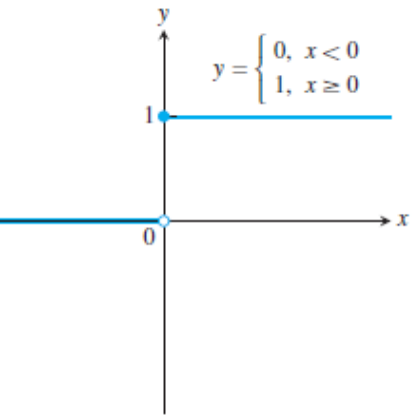
Soldan ve sağdan yaklaşımın farklı

0'ın limiti yok.

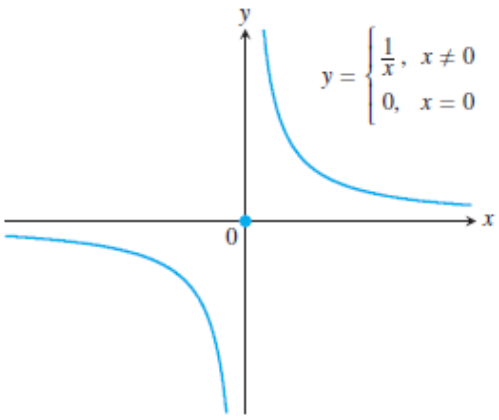
$x \to 0 \neq$

$\lim_{x \to 0} \frac{x^3-8}{x^4-16} = \frac{(x-2)(x^2+2x+4)}{(x^2-4)(x^2+4)}$

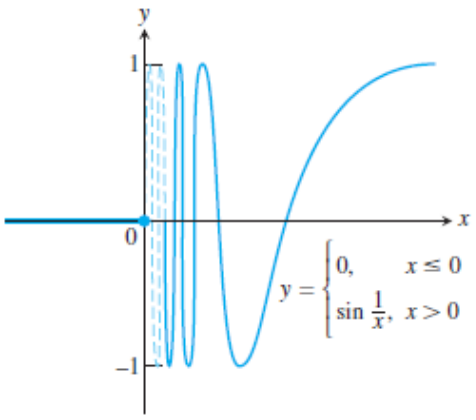
$(x-2)$ $(x+2)$ Birbirini götürür



(a) Unit step function $U(x)$



(b) $g(x)$



(c) $f(x)$

The Limit Laws

THEOREM 1—Limit Laws If L, M, c, and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x \to c}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x \to c}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x \to c}(k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x \to c}[f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(If n is even, we assume that $\lim_{x \to c} f(x) = L > 0$.)

(a) $\lim_{x \to c}(x^3 + 4x^2 - 3)$ **(b)** $\lim_{x \to c} \frac{x^4 + x^2 - 1}{x^2 + 5}$ **(c)** $\lim_{x \to -2} \sqrt{4x^2 - 3}$

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} = \frac{(x+2)(x-1)}{x(x-1)} = \frac{x+2}{x} = \frac{1+2}{1} = \frac{3}{1} = 3$$

Birbirini götürecek

x yerine 1 yazarız.